Tabular Integration

We have seen that integrals of the form $\int f(x)g(x)dx$, in which f can be differentiated repeatedly to become zero and g can be integrated repeatedly without difficulty, are natural candidates for integration by parts. However, if many repetitions are required, the calculations can be cumbersome. In situations like this, there is a way to organize the calculations that saves a great deal of work. It is **tabular integration,** as shown in Examples 6 and 7.

Example 6 USING TABULAR INTEGRATION

Evaluate $\int x^2 e^x \, dx$.

Solution With $f(x) = x^2$ and $g(x) = e^x$, we list:

$f(x)$ and its derivatives		$g(x)$ and its integrals
x^2	(+)	e^x
$2x$	(−)	e^x
2	(+)	e^x
0		e^x

We combine the products of the functions connected by the arrows according to the operation signs above the arrows to obtain

$$\int x^2 e^x \, dx = x^2 e^x - 2xe^x + 2e^x + C.$$

Compare this with the result in Example 4.

Example 7 USING TABULAR INTEGRATION

Evaluate $\int x^3 \sin x \, dx$.

Solution With $f(x) = x^3$ and $g(x) = \sin x$, we list:

$f(x)$ and its derivatives		$g(x)$ and its integrals
x^3	(+)	$\sin x$
$3x^2$	(−)	$-\cos x$
$6x$	(+)	$-\sin x$
6	(−)	$\cos x$
0		$\sin x$

Again we combine the products of the functions connected by the arrows according to the operation signs above the arrows to obtain

$$\int x^3 \sin x \, dx = -x^3 \cos x + 3x^2 \sin x + 6x \cos x - 6 \sin x + C.$$

Section 6.3 Exercises

In Exercises 1–4, evaluate the integral. Confirm your answer by differentiation.

1. $\int x \sin x \, dx$

2. $\int x^2 \cos x \, dx$

3. $\int y \ln y \, dy$

4. $\int \tan^{-1} y \, dy$

In Exercises 5–8, evaluate the integral. Support your answer by superimposing the graph of one of the antiderivatives on a slope field of the integrand.

5. $\int x \sec^2 x \, dx$

6. $\int \sin^{-1} \theta \, d\theta$

7. $\int t^2 \sin t \, dt$

8. $\int t \csc^2 t \, dt$

In Exercises 9–14, evaluate the integral.

9. $\int x^3 \ln x \, dx$

10. $\int x^4 e^{-x} \, dx$

11. $\int (x^2 - 5x)e^x \, dx$

12. $\int x^3 e^{-2x} \, dx$

13. $\int e^y \sin y \, dy$

14. $\int e^{-y} \cos y \, dy$

In Exercises 15–18, evaluate the integral analytically. Support your answer using NINT.

15. $\int_{0}^{\pi/2} x^2 \sin 2x \, dx$

16. $\int_{0}^{\pi/2} x^3 \cos 2x \, dx$

17. $\int_{-2}^{3} e^{2x} \cos 3x \, dx$

18. $\int_{-3}^{2} e^{-2x} \sin 2x \, dx$

In Exercises 19–22, solve the differential equation.

19. $\frac{dy}{dx} = x^2 e^{4x}$

20. $\frac{dy}{dx} = x^2 \ln x$

21. $\frac{dy}{d\theta} = \theta \sec^{-1} \theta, \quad \theta > 1$

22. $\frac{dy}{d\theta} = \theta \sec \theta \tan \theta$

23. *Finding Area* Find the area of the region enclosed by the x-axis and the curve $y = x \sin x$ for

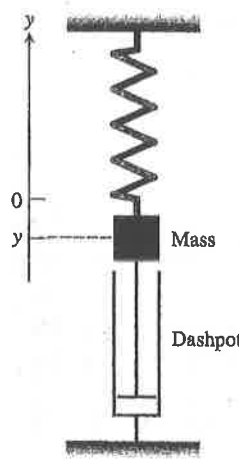
(a) $0 \le x \le \pi$, (b) $\pi \le x \le 2\pi$, (c) $0 \le x \le 2\pi$.

24. *Finding Area* Find the area of the region enclosed by the y-axis and the curves $y = x^2$ and $y = (x^2 + x + 1)e^{-x}$.

25. *Average Value* A retarding force, symbolized by the dashpot in the figure, slows the motion of the weighted spring so that the mass's position at time t is

$$y = 2e^{-t} \cos t, \quad t \ge 0.$$

Find the average value of y over the interval $0 \le t \le 2\pi$.



Exploration

26. Consider the integral $\int x^n e^x \, dx$. Use integration by parts to evaluate the integral if

(a) $n = 1$. (b) $n = 2$. (c) $n = 3$.

(d) Conjecture the value of the integral for any positive integer n.

(e) **Writing to Learn** Give a convincing argument that your conjecture in (d) is true.

In Exercises 27–30, evaluate the integral by using a substitution prior to integration by parts.

27. $\int \sin \sqrt{x} \, dx$

28. $\int e^{\sqrt{3x+9}} \, dx$

29. $\int x^7 e^{x^2} \, dx$

30. $\int \sin (\ln r) \, dr$

In Exercises 31–34, use integration by parts to establish the *reduction formula.*

31. $\int x^n \cos x \, dx = x^n \sin x - n \int x^{n-1} \sin x \, dx$

32. $\int x^n \sin x \, dx = -x^n \cos x + n \int x^{n-1} \cos x \, dx$

33. $\int x^n e^{ax} \, dx = \frac{x^n e^{ax}}{a} - \frac{n}{a} \int x^{n-1} e^{ax} \, dx, \quad a \ne 0$

34. $\int (\ln x)^n \, dx = x(\ln x)^n - n \int (\ln x)^{n-1} \, dx$

Extending the Ideas

35. *Integrating Inverse Functions* Assume that the function f has an inverse.

(a) Show that $\int f^{-1}(x) dx = \int y f'(y) \, dy$. (*Hint:* Use the substitution $y = f^{-1}(x)$.)

(b) Use integration by parts on the second integral in (a) to show that

$$\int f^{-1}(x) \, dx = \int y f'(y) \, dy = x f^{-1}(x) - \int f(y) \, dy.$$

36. *Integrating Inverse Functions* Assume that the function f has an inverse. Use integration by parts directly to show that

$$\int f^{-1}(x) \, dx = x f^{-1}(x) - \int x \left(\frac{d}{dx} f^{-1}(x) \right) dx.$$

In Exercises 37–40, evaluate the integral using

(a) the technique of Exercise 35.

(b) the technique of Exercise 36.

(c) Show that the expressions (with $C = 0$) obtained in parts (a) and (b) are the same.

37. $\int \sin^{-1} x \, dx$

38. $\int \tan^{-1} x \, dx$

39. $\int \cos^{-1} x \, dx$

40. $\int \log_2 x \, dx$